On some numerical computations to the oil-reservoir problems(Mathematical Fluid Mechanics and Modeling)

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On some numerical computations to the oil–reservoir problems

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1. Introduction.

In order to recover part of the remaining oil from wells (called *production wells*), it is used the way that one injects water into another wells (called *injection wells*), which are located around the reservoir, so that the water pushes the oil toward the production wells. In this process, two immiscible fluids — water and oil — flow through the porous medium, and can be regarded as separated by a sharp interface during penetration of water into oil. By laboratory studies, it is already known that the interface is unstable; small perturbations in the interface grow up (see Fig. 1).

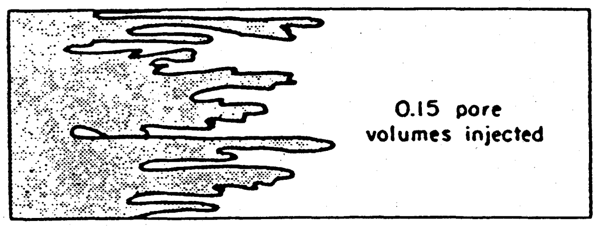

Fig. 1. Instability of the interface [10].

In this paper we show some numerical computations of two-dimensional problems in the following cases.

Case (a). The water is injected into the injection wells;

Case (b). The pressure given at the injection wells is negative.

The oil–recovery problem stated above corresponds to Case (a). When the capillary pressure between two fluid can be neglected, many numerical simulations are done in [4]–[7] and [11]. From the analytical points of view, Chorin [2] proves under some special initial and boundary conditions that the interface is linearly unstable if $\mu = \mu_o/\mu_w > 3$ and is linearly stable if $\mu < 3$, where μ_w and μ_o are the viscosities of water and oil, respectively. To investigate the role of the capillary pressure, we shall show some numerical computations when the capillary pressure is not neglected.

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In Case (b), there exists a steady state solution in one-dimensional problems, and the numerical simulation suggests us that this steady state solution seems to be stable [8]. Here the *steady state solution* means the solution which is independent of time t. Using this solution, we can construct a planar wave solution (see (13)) in a rectangle region . We try to investigate the relationship between the stability of the planar wave solution and the aspect ratio of the rectangle region from numerical points of view. The motivation to this study is as follows. In *another* problems described by some reaction–diffusion equations, it is already proved that the steady state solution of planar type is stable (resp. unstable) when the aspect ratio of a rectangle region is small (resp. large). Our aim is to discuss whether or not the same property holds in the oil–reservoir problems.

In the following section, we show the basic equation and the boundary conditions of our problems. In Section 3, we treat Case (a) from numerical points of view. Some numerical computations in Case (b) are shown in Section 4.

The numerical computations in this paper are made by computers FX/1 and FX/4 in Department of Mathematics, Hiroshima University and workstation Sun4/490 in Information Processing Center, Fukuoka University of Education.

2. Basic equations.

We write water and oil as *fluid* w and *fluid* o, respectively. Let the saturation $s_i(\mathrm{x},t)$ be the fractional amount of fluid i ($i = w, o$) at point x and time t. We denote by $v_i(\mathrm{x},t)$ and $p_i(\mathrm{x},t)$ the velocity and the pressure of fluid i, respectively. Then we have

$$s_w + s_o = 1, \tag{1}$$

$$-\nabla \cdot \rho_i v_i = \frac{\partial}{\partial t}(n\rho_i s_i), \quad i = w, o, \tag{2}$$

$$v_i = -\frac{Kk_i}{\mu_i}(\nabla p_i - \rho_i g), \quad i = w, o, \tag{3}$$

$$p_o - p_w = p_c. \tag{4}$$

Here ρ_i and μ_i are the density and viscosity of fluid i ($i = w, o$), n and K are the porosity and absolute permeability of the medium, and g is the gavity constant. $k_w = k_w(s_o)$ and $k_o = k_o(s_w)$ are the relative permeabilities of fluid w and o, respectively, and $p_c = p_c(s_w)$ is the capillary pressure between two fluids. The forms of these three functions are given by experiments (see [3], for example), and we take

$$k_w(s_o) = (1 - s_o)^2, \quad k_o(s_w) = (1 - s_w)^2, \quad \text{and} \quad p_c(s_w) = \varepsilon(1 - s_w), \tag{5}$$

where ε is some parameter, which means the magnitude of the capillary pressure. The equation (2) expresses the conservation of fluid i, (3) is Darcy's law, which means the flow of fluid i is proportional to the gradient of its pressure. The equation (4) describes the balance law of the pressure.

For simplicity, we assume that ρ_i, μ_i $(i = w, o)$, n and K are constants with respect to x and t, and that there is no external force, that is $g = 0$. Putting $s = s_w$, $v = v_w + v_o$ and $p = p_w$, we can rewrite (1)–(4) as follows.

$$\frac{\partial}{\partial t}s + \nabla \cdot [vf(s)] - \varepsilon \nabla \cdot [d(s)\nabla s] = 0, \tag{6}$$

$$\nabla \cdot v = 0, \tag{7}$$

$$v = -[\lambda(s)\nabla p - \varepsilon\phi(s)\nabla s], \tag{8}$$

where

$$f(s) = \frac{k_w(1-s)}{\lambda(s)}, \quad \lambda(s) = k_w(1-s) + \frac{k_o(s)}{\mu}, \quad \mu = \frac{\mu_o}{\mu_w},$$
$$d(s) = f(s)\phi(s) \quad \text{and} \quad \phi(s) = -\frac{k_o(s)}{\varepsilon\mu}p_c'(s).$$

We consider the two-dimensional problem for (6)–(8) on $\mathrm{x} \equiv (x, y) \in \Omega(\ell) \equiv (0,1)\times(0,\ell)$, where ℓ is positive parameter. Taking Fig. 1 into considerations, we impose the boundary conditions

$$s(0, y, t) = 1, \quad p(0, y, t) = p^* \qquad \text{on} \quad 0 < y < \ell, \quad t > 0, \tag{9}$$

$$s(1, y, t) = 0, \quad p(1, y, t) = 0 \qquad \text{on} \quad 0 < y < \ell, \quad t > 0, \tag{10}$$

$$\frac{\partial}{\partial y}s(x, y, t) = \frac{\partial}{\partial y}p(x, y, t) = 0 \qquad \text{on} \quad (x, y) \in (0, 1) \times \{0, 1\}, \quad t > 0. \tag{11}$$

The initial condition

$$s(x, y, 0) = s_0(x, y) \qquad \text{on} \quad (x, y) \in \Omega(\ell) \tag{12}$$

is also imposed. The conditions (9)–(10) describe the injection of water at $\{0\} \times (0, \ell)$ with the pressure p^*, and the production of oil at $\{1\} \times (0, \ell)$, respectively. (11) shows that $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ are the Neumann boundaries.

Remark 1. Cases (a) and (b) stated in Section 1 correspond to $p^* > 0$ and $p^* < 0$, respectively.

Under the boundary conditions (9)–(11), there is solutions $s(x, y, t)$, $v(x, y, t)$, $p(x, y, t)$ of the following form:

$$s(x, y, t) = \tilde{s}(x, t), \quad v(x, y, t) = \tilde{v}(x, t), \quad p(x, y, t) = \tilde{p}(x, t), \tag{13}$$

where $\tilde{s}(x,t)$, $\tilde{v}(x,t)$ and $\tilde{p}(x,t)$ are the solution of one-dimensional problem of (6)–(8). From one-dimensional version of (7), it follows that $\tilde{v}$ does not depend on x. Using this fact, we find that $\tilde{s}(x,t)$ and $\tilde{v}(t)$ satisfy

$$\tilde{s}_t + \tilde{v} f(\tilde{s})_x - \varepsilon \left(d(\tilde{s})\tilde{s}_x\right)_x = 0 \qquad \text{on} \quad x \in (0,1), \quad t > 0, \tag{14}$$

$$\tilde{v} = (p^* - c\varepsilon)\left(\int_0^1 \frac{dx}{\lambda(\tilde{s})}\right)^{-1} \qquad \text{on} \quad t > 0, \tag{15}$$

$$\tilde{s}(0,t) = 1 \quad \text{and} \quad \tilde{s}(1,t) = 0 \qquad \text{on} \quad t > 0, \tag{16}$$

$$\tilde{s}(x,0) = \tilde{s}_0(x) \qquad \text{on} \quad x \in (0,1), \tag{17}$$

where $c = \displaystyle\int_0^1 \frac{\phi(\sigma)}{\lambda(\sigma)} d\sigma$ is some positive constant.

Remark 2. In one-dimensional problem (14)–(17), the unknown value $\tilde{p}$ can be eliminated.

Remark 3. The solution of the form (13) is called *planar wave solution* or *solution of planar type*.

3. Numerical computations in Case (a).

In this section we review some numerical simulations to the two-dimensional problems in Case (a) (see [8]). Throughout this section we set $p^* = 1$, and the aspect ratio of the region $\Omega(\ell)$ is fixed to $\ell = 1$. When the capillary pressure is negligible, that is $\varepsilon = 0$, there appears a shock–interface between the region where $s = 0$ (oil region) and the region where $s > 0$ (the region where water is penetrated). Figs. 2 and 3 show graphs of the interfaces when $\mu = 0.5$ and $\mu = 20$, respectively. Fig. 4 is the numerical solution s when $\mu = 20$.

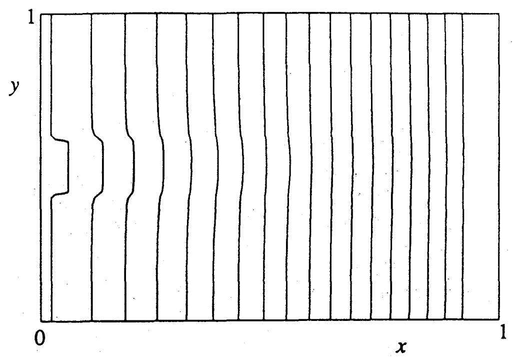


Fig. 2. Numerical interfaces when $\varepsilon = 0$ and $\mu = 0.5$. $t = 0, 0.04, \ldots, 0.68$.

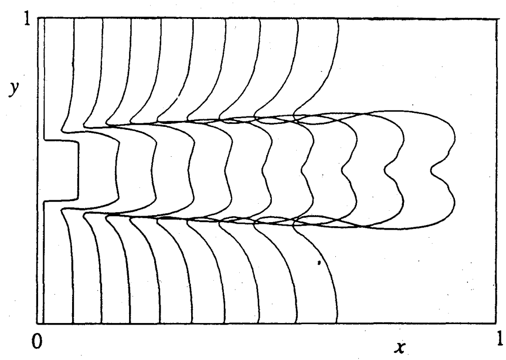


Fig. 3. Numerical interfaces when $\varepsilon = 0$ and $\mu = 20$. $t = 0, 0.4, \cdots, 3.6$.

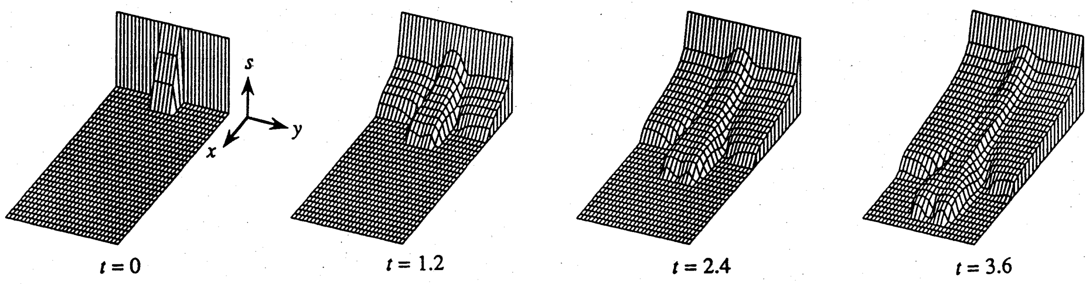


Fig. 4. Numerical simulation when $\varepsilon = 0$ and $\mu = 20$.

From these figures, one can find that the interface seems to be stable when $\mu = 0.5$. However, when $\mu = 20$, the interface is unstable. In one-dimensional problems, we already know the following theorem.

Theorem 4.[8] *Let the initial function be $\tilde{s}_0 \equiv 0$. Then the interface $\eta(t)$ of the solution of (14)–(17) satisfies*

$$\eta''(t) < 0 \quad on \quad 0 < t < T_b \qquad if \quad 0 < \mu < \mu^*, \tag{18}$$

$$\eta''(t) > 0 \quad on \quad 0 < t < T_b \qquad if \quad \mu > \mu^*, \tag{19}$$

where T_b is some positive constant satisfying $\eta(T_b) = 1$, and $\mu^* = 1.65\cdots$.

Remark 5. The constant T_b in the previous theorem is called the *breakthrough time*, which means the time when water reaches to the production wells.

To investigate the role of the capillary pressure, let us consider the case of $\varepsilon > 0$. In the following we set $\mu = 20$; the case where the interface is unstable when $\varepsilon = 0$. For small value of ε, the interface is also unstable (see Fig. 5).

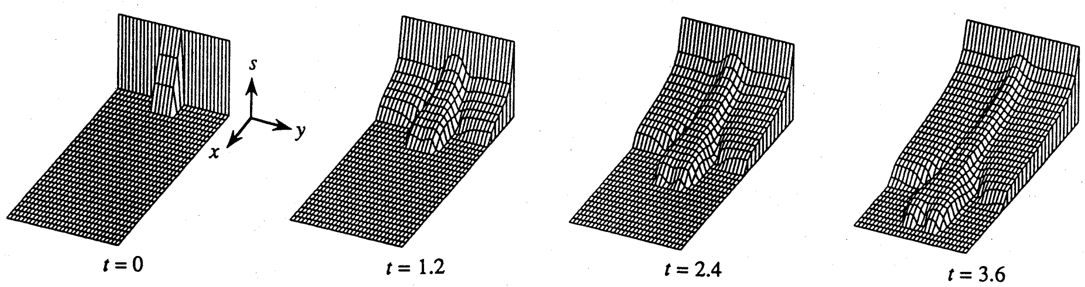


Fig. 5. Numerical simulation when $\varepsilon = 0.01$ and $\mu = 20$.

However for large value of ε, the interface becomes stable.

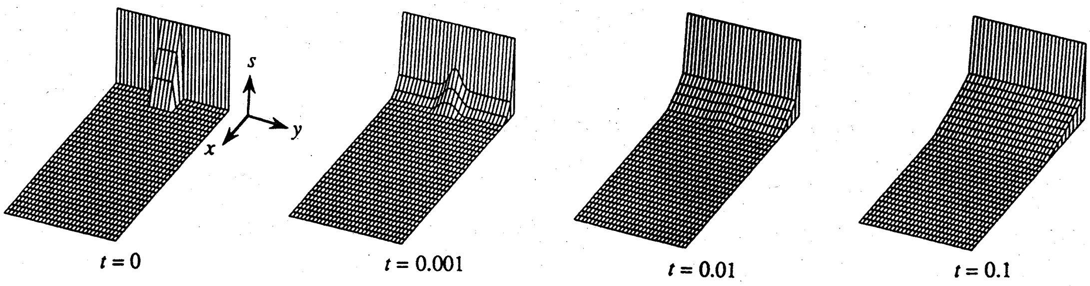


Fig. 6. Numerical simulation when $\varepsilon = 100$ and $\mu = 20$.

Hence one may say that the capillary pressure stabilizes the interface (see [9]).

In one-dimensional problems, by numerical simulations, it seems that the acceleration of the interface is positive for small value of ε and negative for large ε. Hence the capillary pressure reduces the acceleration of the interface.

4. Numerical computations in Case (b).

In this section we discuss the case of $p^* < 0$. In one-dimensional problems, we already know the following theorem.

Theorem 6. [8] *Let $p^* < 0$ and $\varepsilon > 0$. Then there exists a steady state solution $\tilde{s}(x)$ and $\tilde{v}$ of (14)–(17).*

By a numerical simulation (Fig. 7), it seems that the steady state solution is stable; however we can not prove it yet. Throughout this section, we set $\mu = 20$, $p^* = -50$ and $\varepsilon = 100$. The following figure displays the numerical simulation of one-dimensional problem.

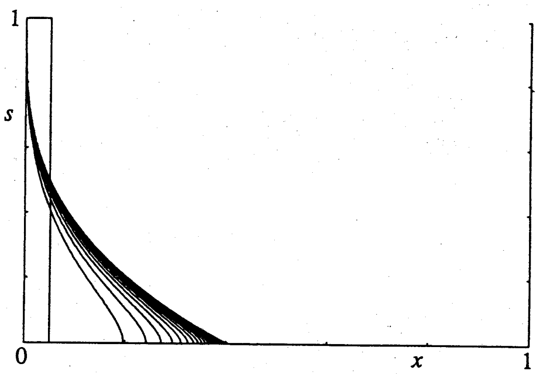


Fig. 7. One-dimensional numerical simulation in Case (b) at $t = 0, 0.005, 0.01, \cdots, 0.5$.

By simple calculations, we find that the steady state solution $\tilde{s}(x)$ satisfies

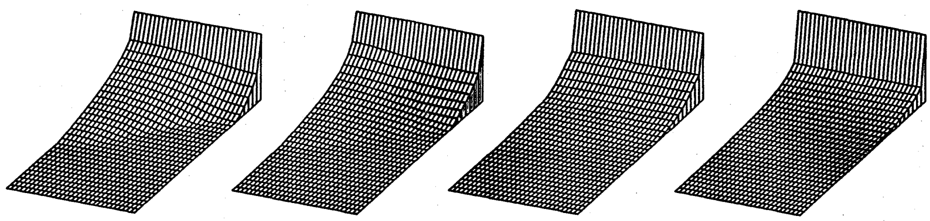
$$\tilde{s}(x) = \begin{cases} 1 - (x/\eta)^{1/3}, & \text{if } \ 0 \le x \le \eta, \\ 0, & \text{if } \ \eta < x \le 1, \end{cases} \tag{20}$$

where $\eta = 0.4$ is the position of the interface. Let us construct the two-dimensional steady state solution of planar type by (13), and we investigate the stability of this solution from numerical points of view. We take the initial function $s_0(x, y)$ as

$$s_0(x, y) = \begin{cases} 1 - (x/\eta(y))^{1/3}, & \text{if } \ 0 \le x \le \eta(y), \quad 0 \le y \le \ell, \\ 0, & \text{if } \ \eta(y) < x \le 1, \quad 0 \le y \le \ell, \end{cases} \tag{21}$$

where $\eta(y) = 0.4 + 0.2 \cos \dfrac{\pi}{\ell} y$.

The following figure demonstrates numerical simulation when $\ell = 0.01$.

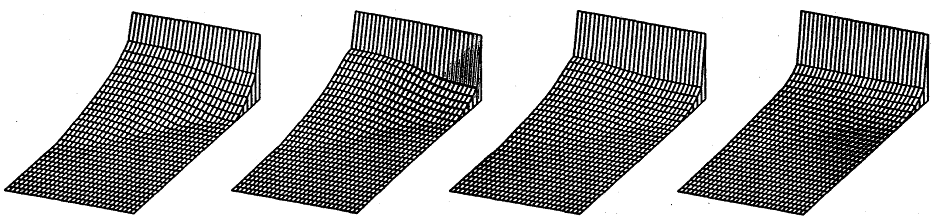


$t = 0$ $\quad$ $t = 0.0001$ $\quad$ $t = 0.001$ $\quad$ $t = 0.01$

Fig. 8. Numerical simulation in Case (b) when $\ell = 0.01$.

One can see that the steady state solution of planar type is stable in this case.

Next we show the case where $\ell = 1000$.



$t = 0$ $\quad$ $t = 0.005$ $\quad$ $t = 0.015$ $\quad$ $t = 0.05$

Fig. 9. Numerical simulation in Case (b) when $\ell = 1000$.

In this case the steady state solution of planar type also seems to be stable. We carry on computations when $\ell = 0.1, 1, 10, 100$. In all cases, we observe that the solutions seem to be stable.

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